

On Groups G_n^2 and Coxeter Groups

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January 1, 2016

Abstract

In the present paper, we prove that the group G_n^2 of free n -strand braids is isomorphic to a subgroup of a semidirect product of some Coxeter group that we denote by $C(n, 2)$ and the symmetric group S_n .

In [4], groups G_n^k closely related to braid groups and various problems in geometry, topology, and dynamical systems were constructed. Many invariants of topological objects are valued in groups G_n^k , hence it is extremely important to solve the word problem and the conjugacy problem for G_n^k .

The first non-trivial instance of the groups G_n^k is the group G_n^2 ; deeper understanding of G_n^2 may allow one to shed light to G_n^k for larger k since there are various homomorphisms $G_n^k \rightarrow G_{n-1}^k$ and $G_n^k \rightarrow G_{n-1}^{k-1}$ finally leading to various homomorphisms $G_n^k \rightarrow G_m^2$ for various $m < n$ [4, 6].

The aim of the present note is to prove that some finite index subgroup of the group G_n^2 is isomorphic to a finite index subgroup of some Coxeter groups of a graph on $\binom{n}{2}$ vertices.

This leads to an algebraic solution of the word problem for the groups G_n^2 . The final result can be formulated in a form of Theorem 3 that G_n^2 embeds into a semidirect product of some Coxeter group $C(n, 2)$ and the permutation group, nevertheless, we first describe it in the term of the “rewriting procedure” which allows one to represent words from Coxeter groups of a special type in terms of G_n^2 , the latter having intrinsic geometrical meaning [4, 7, 5, 10].

The question whether the groups G_n^k for $k > 2$ admit a similar description in terms of Coxeter groups, still remains open.

Let Γ be the graph on $\binom{n}{2}$ vertices whose vertices are denoted by letters b_{ij} , where i, j run all unordered pairs of unequal integers from $\{1, \dots, n\}$. We shall connect two vertices by an edge if they share an index, thus, b_{ij} is connected to b_{jk} for each three distinct letters i, j, k . Let us mark such edges with 3, thus, the corresponding Coxeter group $C(n, 2)$ will be given by presentation

$$\langle b_{ij} | (1), (2), (3) \rangle,$$

where the three groups of relations are

$$(b_{ij}b_{ik})^3 = 1, \forall i, j, k \in \{1, \dots, n\}, \text{Card}(\{i, j, k\}) = 3, \quad (1)$$

$$b_{ij}b_{kl} = b_{kl}b_{ij}, \forall i, j, k, l \in \{1, \dots, n\} : \text{Card}(\{i, j, k, l\}) = 4 \quad (2)$$

$$b_{ij}^2 = 1, \forall i, j \in \{1, \dots, n\}, i \neq j. \quad (3)$$

The groups G_n^2 are known as *free braid groups* [4]; they are also known under other names (*e.g.*, *virtual Gauss braids*), see, e.g. [1, 2].

For an integer $n > 2$, we define the group G_n^2 as the group having the following $\binom{n}{2}$ generators a_m , where m runs the set of all unordered k -tuples m_1, \dots, m_k , whereas each m_i are pairwise distinct numbers from $\{1, \dots, n\}$.

$$G_n^k = \langle a_m | (1'), (2'), (3') \rangle.$$

Here the defining relations look as follows:

$$a_{ij}a_{ik}a_{jk} = a_{jk}a_{ik}a_{ij} \quad (1')$$

for each three distinct indices $i, j, k \in \{1, \dots, n\}$.

$$a_m a_{m'} = a_{m'} a_m, \quad (2')$$

where m and m' are two disjoint 2-element subsets of $\{1, \dots, n\}$.

Finally, for any distinct $i, j \in \{1, \dots, n\}$ we set

$$a_{ij}^2 = 1. \quad (3')$$

Let $l : G_n^2 \rightarrow S_n$ be the homomorphism from G_n^2 to the symmetric group taking each a_{ij} to the transposition (i, j) .

Note that there is a similar homomorphism $m : C(n, 2) \rightarrow S(n)$ which just takes b_{ij} to the transposition (i, j) . As we shall see later, these homomorphisms are closely related to each other.

We say that an element β of G_n^2 is *pure* if $l(\beta) = 1$. Hence, we get a normal subgroup $PG_n^2 = \text{Ker}(l)$ of the group G_n^2 . This normal subgroup is similar to the subgroup of pure braids in the group of all braids.

Let $w = a_{i_1,1}, a_{i_1,2}, \dots, a_{i_k,1}, a_{i_k,2}$ be a word representing an element from G_n^2 . For $j = 1, \dots, k$ by w_j we denote the product of the first j letters of w , w_0 being the empty word. For $p = 1, \dots, k$, we define the permutation $\sigma_p = l(w_p)^{-1}$ with $\sigma_0 = id$. In other words, σ_0 is the identical permutation, and each consequent permutation σ_{j+1} is obtained by multiplying σ_j with the transposition corresponding to w_j on the left.

Now we set

$$\tilde{w} = b_{\sigma_0(i_{1,1}), \sigma_0(i_{1,2})} b_{\sigma_1(i_{2,1}), \sigma_1(i_{2,2})} \cdots b_{\sigma_{k-1}(i_{k,1}), \sigma_{k-1}(i_{k,2})}.$$

This rewriting rule is related to the following two approaches of strand enumeration *the local one* and *the global one*.

Consider the permutation group $S_3 = G_3^2$. View Fig. 1.

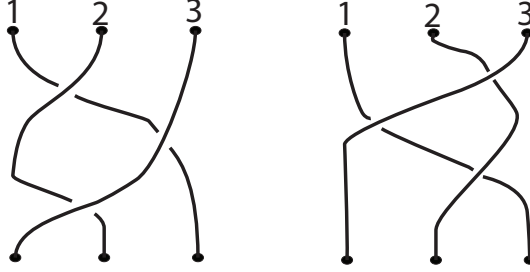


Figure 1: Local enumeration yields $\sigma_1\sigma_2\sigma_2 = \sigma_2\sigma_1\sigma_2$; global enumeration yields $a_{12}a_{13}a_{23} = a_{23}a_{13}a_{12}$

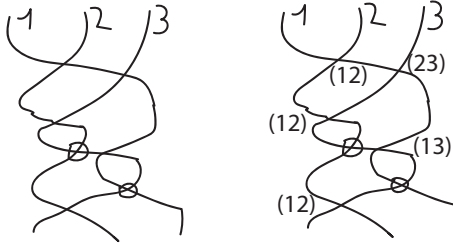


Figure 2: An example of rewriting

If we denote the crossings in a standard way (use Artin's notation), we get the standard Artin's relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$; if we say that i -th crossing σ_i is formed by the intersection of i -th and $(i+1)$ -th strands, we can even rewrite it as $b_{12}b_{23}b_{12} = b_{23}b_{12}b_{23}$ taking $b_{i,i+1}$ for σ_i .

However, if we enumerate the strands according to their upper ends and use a_{ij} for the crossing with strands i and j , we get the standard relation $a_{12}a_{13}a_{23} = a_{23}a_{13}a_{12}$ for G_n^2 .

Example. Consider the case $n = 3, w = a_{12}a_{13}a_{23}a_{13}a_{23}$. Then we have the following permutations: $\sigma_0 = id, \sigma_1 = (12), \sigma_2 = (132), \sigma_3 = id, \sigma_4 = (13), \sigma_5 = (213)$. Thus, we get the corresponding word $\tilde{w} = b_{12}b_{23}b_{12}b_{13}b_{12}$, see Fig. 2.

Theorem 1. *If two words w, w' generate two equal elements of G_n^2 then the words \tilde{w}, \tilde{w}' generated equal elements of $C(n, 2)$.*

Moreover, $l(w) = m(\tilde{w})^{-1}$.

Proof. It suffices for us to check the following three cases: when w' is obtained from w by an addition/removal of a pair of equal subsequent letters, when w' is obtained from w by applying the far commutativity relation $a_{ij}a_{kl} = a_{kl}a_{ij}$ (for all i, j, k, l distinct) and when w' is obtained from w by obtaining (1').

In the first case, at some step p we have some two letters $a_{ij}a_{ij}$ and a permutation σ_p ; thus, instead of the letter a_{ij} we get the letter $b_{\sigma_p(i)\sigma_p(j)}$. However, $\sigma_{p+1}(i) = \sigma_p(j), \sigma_{p+1}(j) = \sigma_p(i)$. Besides that, $\sigma_{p+2} \equiv \sigma_p$. Consequently, the addition/removal of two subsequent equal letters $w \rightarrow w'$ yields an addition/removal of two subsequent equal letters $\tilde{w} \rightarrow \tilde{w}'$.

If in w at p -th step we have two commuting letters $a_{ij}a_{kl}$ then the corresponding letters in w' will commute as well: the value of the permutation σ on k, l does not depend on the value of this

permutation on i, j and does not change when the corresponding transposition is applied.

Let us consider the most interesting case. Assume the letters in w in the positions $p+1, p+2, p+3$ are $a_{ij}a_{ik}a_{jk}$. Then we have: $\sigma_{p+1}(j) = \sigma_p(i), \sigma_{p+1}(i) = \sigma_p(j)$. Hence, $\sigma_{p+2}(i) = \sigma_{p+1}(k) = \sigma_p(k); \sigma_{p+2}(k) = \sigma_{p+1}(i) = \sigma_p(j), \sigma_{p+2}(j) = \sigma_{p+1}(j) = \sigma_p(i)$. Finally, $\sigma_{p+3}(i) = \sigma_{p+2}(i) = \sigma_p(k); \sigma_{p+3}(j) = \sigma_{p+2}(k) = \sigma_p(j), \sigma_{p+3}(k) = \sigma_{p+2}(j) = \sigma_p(i)$.

Thus, the corresponding three letters in the word \tilde{w} will look like $b_{\sigma_p(i)\sigma_p(j)}b_{\sigma_p(j)\sigma_p(k)}b_{\sigma_p(i)\sigma_p(j)}$.

In the same way, we check that in the word w' the images of the letters $a_{jk}a_{ik}a_{ij}$ will look like $b_{\sigma_p(j)\sigma_p(k)}b_{\sigma_p(i)\sigma_p(j)}b_{\sigma_p(j)\sigma_p(k)}$ and the permutation after these three letters will be the same:

$$\sigma_{p+3}(i) = \sigma_p(k); \sigma_{p+3}(j) = \sigma_p(j), \sigma_{p+3}(k) = \sigma_p(i).$$

Since the relation $b_{\sigma_p(i)\sigma_p(j)}b_{\sigma_p(j)\sigma_p(k)}b_{\sigma_p(i)\sigma_p(j)} = b_{\sigma_p(j)\sigma_p(k)}b_{\sigma_p(i)\sigma_p(j)}b_{\sigma_p(j)\sigma_p(k)}$ holds in $C(n, 2)$, we get the desired statement.

The last statement of the theorem can be reformulated as follows: the product of transpositions corresponding to the word w read from the right to the left corresponds to the product of transpositions corresponding to \tilde{w} read from the left to the right.

This statement can be proved by induction: we start with the transposition $(i_{1,1}, i_{j,1})$ for both w and \tilde{w} and then note that $(ab)^{-1} = a^{-1}(ab^{-1}a^{-1})$.

□

Denote the map $PG_n^2 \rightarrow C(n, 2) : w \rightarrow \tilde{w}$, we have constructed, by *co*.

It is well known that for Coxeter groups there exist a gradient descent algorithm which can means the following. Given a Coxeter group W with a standard system of generators $S = \{s_i\}$ and relations $s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1$. Rewrite the latter relations as $s_i s_j \cdots = s_j s_i \cdots$ (m_{ij} factors on the LHS and m_{ij} factors on the RHS). This exchange relation is an *elementary equivalence* which does not change the length. Besides that we have the two the elementary equivalence $s_j s_j = 1$ which decreases (increases) the length by 2.

We say that a word w in generators s_i is *reduced* if its length is minimal among all words representing the same element of W . It is known that from each word w one can get to any reduced word w' representing the same element of W by using only elementary equivalences not increasing the length. In particular, any for any two reduced words w, w' representing the same element of W , we can get from w to w' by a sequence of exchanges.

From Theorem 1 immediately see that the same is true for G_n^2 , namely, we get the following

Corollary 1. *For each word w in a_{ij} and a reduced word w' equivalent to it in G_n^2 one can get from w to w' by a sequence of far commutativity relations (2'), exchanges of the type $a_{ij}a_{ik}a_{jk} \rightarrow a_{jk}a_{ik}a_{ij}$ (1'), and cancellations of two identical letters $a_{ij}a_{ij} \rightarrow \emptyset$.*

In particular, if both w and w' are reduced then we can get from w to w' by using only (1') and (2').

The inverse map to *co* is constructed in a similar way: one should just take into account that if \tilde{w} corresponds to w then $l(w) = m(\tilde{w})^{-1}$. This means that having the word \tilde{w} , we know all permutations $\sigma_k(w)$ and $l(w)$ for the potential word w we are constructing.

The above arguments lead us to the following

Theorem 2. *The map $co : w \mapsto \tilde{w}$ is an isomorphism $PG_n^k \rightarrow' C(n, 2)$.*

Proof. Indeed, the fact that equivalent words are mapped to equivalent words is proved in Theorem 1. Taking into account that for words from PG_n^2 the permutation $l(w)$ is trivial, we see that for $w_1, w_2 \in PG_n^2$ the equality $(w_1 \tilde{w}_2) = \tilde{w}_1 \tilde{w}_2$ takes place; hence the map co is a homomorphism. The existence of the inverse homomorphism is proved analogously. \square

The rewriting techniques can be formulated as follows. Consider the semidirect products $G_n^2 \rtimes S_n$ and $C(n, 2) \rtimes S_n$ where the permutation group S_n acts on the generators a_{ij} (resp., b_{ij}) by permutations of indices.

Theorem 3. *There is an isomorphism between semidirect products $G_n^2 \rtimes S_n$ and $C(n, 2) \rtimes S_n$ which takes $(b_{ij}, 1)$ to $(a_{ij}, (ij))$ for each transposition (ij) and $(\sigma, 1) \rightarrow (\sigma, 1)$.*

Hence, G_n^2 is a normal subgroup of $C(n, 2) \rtimes S_n$.

1 Acknowledgements

I am grateful to L.A.Bokut' and E.B. Vinberg for useful discussions; I am especially grateful to E.B.Vinberg who suggested the formulation of Theorem 3.

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